Deriving an Explicit Polynomial Counterexample to the Jacobian Conjecture

A reproducible cubic-factor and resultant construction

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Abstract

This note gives a self-contained derivation of a polynomial map $F\colon \mathbb{C}^3 \to \mathbb{C}^3$ with $\det JF = -2$ and a three-point fiber. The construction starts from the three choices of a linear factor of a cubic. Fixing one cubic coefficient and fixing the resultant of the linear and quadratic factors removes finite ramification; an explicit polynomial-coordinate calculation identifies the resulting threefold with affine 3-space. Expanding the coefficient map then gives the example. The emphasis is on the mathematical motivation, the decisive coordinate choices, and exact Jacobian and fiber computations.

Scope of the derivation. This is a concise, reproducible mathematical reasoning summary, not a transcript of private internal chain-of-thought. All decisive identities, choices, and motivations needed to reconstruct and check the example are included explicitly.

Contents

1 Motivation: what a counterexample would have to do

A polynomial map with nonzero constant Jacobian is *étale*: it has no finite critical point and is locally an algebraic isomorphism. It is consequently quasi-finite and open. These elementary facts sharply constrain any noninvertible example.

First, such a map cannot omit a target hypersurface. If an irreducible polynomial q vanished on an omitted divisor, then $q(F)$ would be a nowhere-zero polynomial on $\mathbb{C}^n$, hence a nonzero constant, contradicting dominance. Thus the omitted locus has codimension at least 2.

Second, a noninvertible example cannot be finite. A finite *étale* cover of $\mathbb{A}^n_{\mathbb{C}}$ is trivial, so the possible failure is nonproperness: over a target divisor some sheets must escape to infinity while at least one unramified sheet remains visible. In the finite normalization of the function-field extension, the corresponding generic inertia permutation must have a fixed sheet.

This makes degree 3 the first natural target for the construction. In degree 2, the only nontrivial inertia permutation is a transposition and it fixes no sheet. In degree 3, a transposition has cycle type $(2,1)$: two sheets can disappear at infinity while one simple sheet remains. The generic monodromy can be S_3, so a Galois-cover obstruction is avoided. This is the motivating sheet-count calculation; the construction below does not rely on any classification of Keller maps.

There is a further affine-space constraint. Simply deleting a *principal* ramification divisor $V(r)$ from a finite cover makes r a nonconstant unit on the resulting open variety; that open cannot be $\mathbb{A}^n$, whose only units are constants. This rules out the most immediate monogenic or hypersurface-cover constructions and motivates a nonmonogenic, degree-3 construction in which the ramification is absorbed by a normalization condition rather than by inverting a polynomial coordinate.

The useful mechanism is to factor a cubic. A generic cubic has three choices of a linear factor, the common-root condition is measured by a resultant, and the factor-rescaling action changes that resultant with weight one. Fixing the resultant to 1 selects a global scale for each coprime factorization and removes finite ramification. The remaining task is to find an affine 3-space chart on the resultant-normalized factor space.

The construction strategy is three sheets $\longrightarrow$ cubic factorization $\longrightarrow$ resultant slice $\longrightarrow$ an affine chart $\longrightarrow$ an *étale* polynomial map.

2 The guiding construction: factor a cubic

Take a linear and a quadratic polynomial

$$L(t) = xt + \beta, \qquad Q(t) = \gamma t^2 + \delta t + \varepsilon.$$

Their product is

$$L(t)Q(t) = c_3 t^3 + c_2 t^2 + c_1 t + c_0, \tag{1}$$

with

$$c_3 = x\gamma, \qquad c_2 = x\delta + \beta\gamma, \qquad c_1 = x\varepsilon + \beta\delta, \qquad c_0 = \beta\varepsilon. \tag{2}$$

The resultant is

$$\rho = \operatorname{Res}(L, Q) = x^2\varepsilon - x\beta\delta + \beta^2\gamma. \tag{3}$$

The following elementary calculation explains why the resultant is the right normalization.

Lemma 1 (Coefficient–resultant Jacobian)**.** *One has the polynomial identity*

$$\det \frac{\partial(c_3, c_2, c_1, c_0, \rho)}{\partial(x, \beta, \gamma, \delta, \varepsilon)} = -\rho^2. \tag{4}$$

Proof. Differentiate the five expressions in (2)–(3) and expand the resulting 5×5 determinant. The factor is exactly

$$-(\beta^2\gamma - x\beta\delta + x^2\varepsilon)^2 = -\rho^2.$$

This is a polynomial identity; no localization or genericity assumption is involved. □

Thus the multiplication map is nonsingular whenever the factors are coprime. Fix the coefficient and resultant

$$c_2 = 1, \qquad \rho = 1. \tag{5}$$

The ambiguity

$$(L, Q) \longmapsto (\lambda L, \lambda^{-1} Q)$$

preserves the product and satisfies

$$\operatorname{Res}(\lambda L, \lambda^{-1} Q) = \lambda \operatorname{Res}(L, Q).$$

Consequently every generic choice of linear factor has a unique scaling for which $\rho = 1$. This predicts a generically three-to-one coefficient map.

Using $x\delta + \beta\gamma = 1$, the resultant equation becomes

$$1 = x^2\varepsilon - x\beta\delta + \beta^2\gamma = x^2\varepsilon - \beta + 2\gamma\beta^2.$$

Hence the slice is

$$Y = \begin{Bmatrix} x\delta + \beta\gamma = 1, \\ x^2\varepsilon - \beta + 2\gamma\beta^2 = 1 \end{Bmatrix} \subset \mathbb{A}^5_{x,\beta,\gamma,\delta,\varepsilon}. \tag{6}$$

Lemma 1 shows that the coefficient map restricted to Y is everywhere *étale*. The decisive step is now to identify Y with $\mathbb{A}^3$.

3 Finding the polynomial affine chart

The first coordinate is forced by the equations of Y. Substitute $\beta\gamma = 1 - x\delta$ into the second equation of (6):

$$1 = x^2\varepsilon - \beta + 2\beta(1 - x\delta) = x^2\varepsilon + \beta - 2x\beta\delta.$$

Therefore

$$\beta - 1 = x(2\beta\delta - x\varepsilon). \tag{7}$$

It is natural to set

$$y = 2\beta\delta - x\varepsilon, \qquad \text{so that} \qquad \beta = 1 + xy.$$

To solve the second equation polynomially, one needs $1 + \beta - 2\gamma\beta^2$ to be divisible by x^2. Its linear term is canceled precisely by taking

$$\gamma = 1 - \frac{3}{2}xy + \frac{1}{2}x^2 z,$$

where z is the remaining free coordinate. Solving successively for δ and ε then yields a completely polynomial chart.

Proposition 2 (Affine-space parametrization). *The map $\mathbb{A}^3_{x,y,z} \to Y$ defined by*

$$\boxed{\begin{aligned} &\beta = 1 + xy, \\ &\gamma = \frac{2 - 3xy + x^2 z}{2}, \\ &\delta = -\frac{xz - y + x^2yz - 3xy^2}{2}, \\ &\varepsilon = -z(1 + xy)^2 + 4y^2 + 3xy^3 \end{aligned}} \tag{8}$$

is a polynomial isomorphism $\mathbb{A}^3 \cong Y$.

Proof. Direct substitution gives the exact identities

$$x\delta + \beta\gamma = 1, \qquad x^2\varepsilon - \beta + 2\gamma\beta^2 = 1. \tag{9}$$

There are no omitted denominators: the only divisions in (8) are by the constant 2.

For completeness, the inverse has a particularly simple polynomial form:

$$y = 2\beta\delta - x\varepsilon \tag{10}$$

and

$$z = 4y\delta + 2\gamma y^2 - \varepsilon. \tag{11}$$

Substitution of (8) into (10)–(11) recovers y, z identically. Conversely the two equations of Y reduce the four coordinate differences to zero, so both compositions are the identity. More explicitly, (7) gives $\beta = 1 + xy$, the first slice equation gives

$$\gamma - 1 = -x(\delta + \gamma y),$$

and combining the two slice equations gives

$$2\gamma - 2 + 3xy = x^2(4y\delta + 2\gamma y^2 - \varepsilon) = x^2 z.$$

These identities recover γ, and then the slice equations recover the displayed δ and ε, without ever dividing by x. □

The coefficient $-\frac{3}{2}$ in γ is not a guess without purpose: it is the cancellation required to make the resultant equation solve polynomially at $x = 0$. This is exactly where a naive rational parametrization would otherwise introduce a forbidden denominator.

4 Expanding the coefficient map

On Y, retain the three free coefficients and rescale the first two outputs:

$$F = (2c_3, 2c_1, c_0) = \big(2x\gamma,\ 2(x\varepsilon + \beta\delta),\ \beta\varepsilon\big). \tag{12}$$

Substitution of (8) gives the polynomial map.

Theorem 3 (Explicit map)**.** *The construction above gives*

$$\boxed{\begin{aligned} F_1 &= x^3z - 3x^2y + 2x, \\ F_2 &= -3x^3y^2z + 9x^2y^3 - 6x^2yz + 12xy^2 - 3xz + y, \\ F_3 &= -x^3y^3z + 3x^2y^4 - 3x^2y^2z + 7xy^3 - 3xyz + 4y^2 - z. \end{aligned}} \tag{13}$$

It satisfies $\det JF = -2$.

Proof. The expansion follows directly from (8) and (12). On the dense open set $x \neq 0$, the equations of Y give

$$\delta = \frac{1 - \beta\gamma}{x}, \qquad \varepsilon = \frac{1 + \beta - 2\gamma\beta^2}{x^2},$$

so

$$F = \left(2x\gamma, \frac{2(1 + 2\beta - 3\gamma\beta^2)}{x}, \frac{\beta + \beta^2 - 2\gamma\beta^3}{x^2}\right). \tag{14}$$

An exact 3×3 determinant expansion gives

$$\det \frac{\partial(F_1, F_2, F_3)}{\partial(x, \beta, \gamma)} = -\frac{4}{x^3}. \tag{15}$$

Indeed, the scalar numerator after expansion is

$$\begin{aligned} \gamma(3\beta^2 + 2\beta^3 - 6\gamma\beta^4) - (1 + 2\beta - 3\gamma\beta^2)(1 + 2\beta - 6\gamma\beta^2) \\ + (4 - 12\gamma\beta)(\beta + \beta^2 - 2\gamma\beta^3) = -1, \end{aligned}$$

so all nonconstant terms cancel.

From (8),

$$\det \frac{\partial(x,\beta,\gamma)}{\partial(x,y,z)} = \det \begin{pmatrix} 1 & 0 & 0 \\ y & x & 0 \\ xz - \frac{3}{2}y & -\frac{3}{2}x & \frac{1}{2}x^2 \end{pmatrix} = \frac{x^3}{2}.$$

The chain rule gives

$$\det JF = -\frac{4}{x^3} \cdot \frac{x^3}{2} = -2$$

on $x \neq 0$. Since F is polynomial, the identity extends across $x = 0$. □

5 The collision and its factorization meaning

At the target $(0, 2, 0)$, the product cubic (1) is

$$\frac{F_1}{2}t^3 + t^2 + \frac{F_2}{2}t + F_3 = t^2 + t = t(t+1).$$

The following three factorizations all have resultant 1:

(x,y,z)	$L(t)$	$Q(t)$
$(-1,2,-8)$	$-t-1$	$-t$
$(0,2,16)$	1	t^2+t
$(1,-1,-5)$	t	$t+1$

They consequently determine three points in the same fiber:

$$\boxed{F(-1,2,-8) = F(0,2,16) = F(1,-1,-5) = (0,2,0).} \tag{16}$$

The factor $L = 1$ is the projective factor at infinity: although the cubic drops in affine degree at this target, its three projective factor choices remain visible.

The fiber can also be checked entirely algebraically. The ideal $(F_1, F_2 - 2, F_3) \subset \mathbb{C}[x,y,z]$ has the triangular form

$$x(x-1)(x+1) = 0, \qquad 3x^2 + 3x + 2y - 4 = 0, \qquad 45x^2 - 3x + 2z - 32 = 0. \tag{17}$$

Thus the fiber consists exactly of the three distinct reduced points in (16). In particular F is not injective, so it has no polynomial left inverse and hence no polynomial inverse.

6 Reproducible derivation summary

The construction can be reproduced from the following short sequence of mathematical decisions and calculations:

1. Use the *étale*/nonproper constraints to target a non-Galois degree-3 construction with a surviving sheet over a branch divisor.

2. Multiply a linear polynomial and a quadratic polynomial, retaining their four coefficients and resultant.

3. Compute the coefficient–resultant Jacobian $-\rho^2$, fix one cubic coefficient, and set $\rho = 1$ to remove finite ramification.

4. Derive $y = 2\beta\delta - x\varepsilon$, hence $\beta = 1 + xy$; cancel the linear obstruction in the resultant equation by taking $\gamma = 1 - \frac{3}{2}xy + \frac{1}{2}x^2 z$, and complete the polynomial affine chart.
5. Expand the remaining coefficient map. The factors x^3 and x^{-3} in the two elementary Jacobians cancel, leaving $\det JF = -2$.
6. Read off a three-point fiber from the three resultant-1 factorizations of $t^2 + t$.

The decisive mechanism is simple: fixing the resultant makes the factor map locally invertible, while the three choices of a linear factor make it globally three-to-one. The affine chart converts that mechanism into explicit coordinate polynomials.